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Research Article

Ecuaciones de ahusamiento y volumen implícito para *Pinus leiophylla* Schiede *ex* Schltdl. & Cham. en Michoacán

Tapering and implied volume equations for *Pinus leiophylla* Schiede *ex* Schltdl. & Cham. in state of *Michoacán*

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Resumen

La descripción del ahusamiento (*di*) y la estimación precisa del volumen fustal (*Vf*) son fundamentales en la estimación de las existencias maderables y la distribución de productos. El objetivo del estudio fue ajustar una función de ahusamiento y definir el modelo de *Vf* implícito para árboles de *Pinus leiophylla* en la comunidad indígena Patambán, Tangancícuaro, Michoacán, México. Con información de 36 árboles dominantes provenientes de plantaciones forestales (245 datos de diámetro normal [*d*], diámetros [*di*] y alturas [*Ai*] a distintas secciones del fuste), se ajustaron ocho funciones de ahusamiento. Mediante Modelos de efectos mixtos (MEM) y la técnica de Máxima verosimilitud se realizó la corrección de heterocedasticidad y autocorrelación, con la expresión *varExp*: $var(\varepsilon_{ij}) = exp^{2 \cdot \delta_1 \cdot v_i}$ y una estructura de media móvil (ARMA) de orden (*p*, *q*), respectivamente. Estadísticamente, la expresión de ahusamiento de *Clutter*: $di = 2.256 \times d^{0.943} \times (At - Ai)^{1.434+ui} \times At^{-1.617}$ fue más precisa al incluir de forma aditiva el efecto aleatorio (ui) en el parámetro relacionado a la altura total (*At*). La explicación de la variabilidad muestral fue de 97.8 %, con un error global de estimación de 1.406 cm y sesgo individual de 0.0138 cm. El modelo de *Vf* implícito fue *Schumacher-Hall*: $Vf = 1.3150 \times At^{0.6345} \times d^{3.8853}$ (*d* y *At* en m). Las expresiones propuestas pueden incluirse de manera confiable en la descripción de datos de inventarios o en las estimaciones de rendimiento para cultivos forestales.

Palabras clave: Aprovechamiento forestal, *Clutter*, efectos mixtos, mercado diferenciado, perfil fustal, *Schumacher-Hall*.

Abstract

The description of the taper (*di*) and the precise estimation of the stem volume (*sV*) are fundamental in the estimation of the timber stock and the distribution of products. The objective of the study was to fit a taper function and define the implicit *sV* model for *Pinus leiophylla* trees in the indigenous community of *Patambán*, *Tangancícuaro*, state of *Michoacán*, Mexico. Eight taper functions were fitted using data from 36 dominant trees from forest plantations (245 normal diameter [*d*], diameters [*di*] and heights [*Ai*] at different stem sections). Mixed effects models (MEM) and the maximum likelihood technique were utilized to correct for heteroscedasticity and self-correlation with the expression *varExp*: $var(\varepsilon_{ij}) = exp^{2\cdot\delta_1\cdot v_i}$ and a moving average (ARMA) structure of order (*p*, *q*), respectively. Statistically, Clutter's taper expression: $di = 2.256 \times d^{0.943} \times (Th - Ih)^{1.434+ui} \times Th^{-1.617}$ was more accurate as it included the random effect (ui) additively in the total height parameter (*Th*). The explanation of sampling variability was 97.8 %, with an overall estimation error of 1.406 cm and an individual bias of 0.0138 cm. The implicit *sV* model was Schumacher-Hall: $Sv = 1.3150 \times Th \times d^{3.8853}$ (*d* and *Th* in m). The proposed expressions can be reliably included in the description of inventory data or yield estimates for forest crops.

Key words: Forest harvesting, Clutter, mixed effects, differentiated market, forest profile, Schumacher-Hall.

Introduction

With the description of the diameter at different heights along the stem (taper: *di*, cm), it is possible to detail the information of an inventory by quantifying the distribution of products and adjusting the timber stock for a differentiated market (Rachid et al., 2014). There are several mathematical expressions in the literature that help describe the *di*. In Mexico, order *n* (Hernández-Ramos et al., 2018; Ramírez-Martínez et al., 2018), exponential (Pompa et al., 2009), and segmented polynomial models (Hernández-Ramos et al., 2017; Tamarit et al., 2014) have been used satisfactorily in different species and under different growth conditions. And so have compatible tapering and trading volume (Cruz-Cobos et al., 2008; García-Espinoza et al., 2019) or volume ratio systems (Cruz-Cobos et al., 2023) as well.

Generally, statistic fit is performed by Ordinary least squares (OLS) (Rachid et al., 2014), seemingly uncorrelated equations (SUEs) (Flores et al., 2021), maximum likelihood (MLL) (Hernández-Ramos et al., 2017), nonlinear generalized least squares (NLGLS) (Monárrez-González et al., 2024), and Marquardt sum of squares minimization algorithm (Niño et al., 2018). However, mixed-effects models (MEM)

(Tamarit et al., 2014) or the inclusion of Dummy variables can statistically improve the results (Torres et al., 2020).

In the stem profile function or simultaneous system of equations commercial-volume-shooting equation fits, data from the same tree are used; then, the error components are correlated, and the OLS fit will produce theoretically robust estimators, but ignoring the errors (Cruz-Cobos et al., 2008; Hernández et al., 2013). The MEM technique is a viable option for solving multicollinearity problems (Pinheiro & Bates, 2000) in any of the three approaches for its application: (I) Prediction of the variable of interest with greater certainty through statistical improvement (Cruz-Cobos et al., 2008; Tamarit et al., 2014); (II) Estimation of the variance components to broaden their applicability or reduce the sampling effort (Saygili & Kahriman, 2023); and (III) Understanding the phenomenon by explaining the genotype-environment interaction (Balzarini, 2002; Bandera & Pérez, 2018).

MEMs incorporate random effects into their parameters, which favorably influences the error term (Correa & Salazar, 2016; Pinheiro et al., 2025), as they correct the variance-covariance structure associated with linear or tree remedy data; this technique has been used to model tree volume and growth patterns, which has generated better results, with respect to OLS (Zuur et al., 2009). In *Michoacán*, forest plantations (FP) are an option to reduce the pressure on forests to meet society's demand for timber. Therefore, there is a need for quantitative silvicultural tools according to the conditions of each species for the management of resources, which has led to contemplate the objective of adjusting a forest profile function and defining the forest volume for trees of *Pinus leiophylla* Schiede *ex* Schltdl. & Cham. in the indigenous community of *Patambán*, *Tangancícuaro* municipality, state of *Michoacán*, Mexico.

Materials and Methods

The study was conducted in the indigenous community of *Patambán*, municipality of *Tangancícuaro*, state of *Michoacán*, Mexico, located in the *Tarascan* Plateau, which is in the physiographic region of the Neovolcanic Axis, belonging to the complex volcanic mountain range and small mountain valleys system (20-65 % slopes) between 1 700 and 3 500 m of altitude. This community is located at an 1 740 masl, at 19°53'45.80" N and 102°12'51.20" W, sub-humid climate with summer rains, an average temperature of 16 to 18 °C, and an Andosol type soil (Instituto Nacional de Estadística y Geografía [INEGI], 2010).

A random sampling of 12 temporary quadrangular sites of 20×20 m (400 m^2) was carried out in four plantations of *P. leiophylla*, ranging in age from 8 to 28 years. Data on tree density (trees ha^{-1}) were collected at each site, and 36 dominant individuals were selected at the site (*i. e.*, three trees per site), whose normal diameter was measured directly at a height of 1.30 m above the ground (*d*, m) with model P. O. BOX JACKSON Forestry Suppliers Inc.® diameter tape; in addition, the diameter at different heights above the stem (*di*) were measured at 0.3, 0.7, 1.30 m and every 2.5 m of height, starting at the diameter and height of the stump. Indirectly, with the model SW Bitterlich® Tele-relascope, until reaching the total height (*Ht*, m), and from the diameter of the stump (*di*=0), the diameter dimensions were determined to subsequently perform the tapering calculations at different heights on the trunk of each individual (*Hi*, m).

The volume per section ($V_{section}$, m^3) was calculated using the Smalian formula (Equation 1) and the tip volume (V_{tip}, m^3), with the cone expression (Equation 2) (Niño et al., 2018). These two volumes ($V_{section}$ and V_{tip}) were added to obtain the stem volume of the tree (*Sv*, m^3).

$$V_{section} = \left(\frac{g_{n-1}+g_i}{2}\right) \times L \qquad (1)$$

$$V_{tip} = \left(\frac{g_n \times L}{3}\right) \qquad (2)$$

Where:

$V_{section}$ = Log volume (m^3)

V_{tip} = Tip volume (m^3)

g_{n-1} = Basal area of the largest diameter of the log (m^2)

g_i = Basal area of the smallest diameter of the log (m^2)

g_n = Basal area of the largest tip diameter (m^2)

L = Log length (m)

Based on 245 pairs of *di* and *Hi* data, eight tapering expressions were fitted (Pompa et al., 2009; Ramírez-Martínez et al., 2018; Torres et al., 2020) using the Rstudio® 2024.04.2 version Build 764 software, in a first approach through nonlinear least squares (NLS) with the *nls* function (Table 1) (Baty et al., 2015; R Core Team, 2024). To avoid statistical convergence issues in the fits, a value of *delta*=0.01 and a *tH* value equal to zero were included in the *di* to prevent loss of observations (Hernández et al., 2013).

Table 1. Tapering Equations used for the fit.

ID	Name	Expression
(3)	Clutter	$di = \beta_1 d^{\beta_2}(Ht - Hi)^{\beta_3} Ht^{\beta_4} + \varepsilon$
(4)	*Cielito* 1	$di = d \times \left[\beta_1 \times \frac{Ht - Hi}{Ht} + \beta_2 \times \left(\frac{Ht - Hi}{Ht}\right)^2 + \beta_3 \times \left(\frac{Ht - Hi}{Ht}\right)^3\right]^{\frac{1}{2}} + \varepsilon$
(5)	*González*	$di = \beta_1 + \beta_2 d \times \left(1 - \frac{Hi}{tH}\right) + \varepsilon$
(6)	Amidon	$di = \beta_1 + d + \left(\frac{Ht - Hi}{Ht - 1.3}\right) + \beta_2 \times \frac{(Ht^2 - Hi^2)(Hi - 1.3)}{Ht^2} + \varepsilon$
(7)	*Rentería*	$di = d \times \sqrt{\beta_1 \times \left(\frac{Ht - Hi}{Ht}\right) + \beta_2 \times \left(\frac{Ht - Hi}{Ht}\right)^2} + \beta_3 \times \left(\frac{Ht - Hi}{Ht}\right)^3 + \varepsilon$
(8)	Forslund	$di = d \times \left(1 - \left(\frac{Hi}{Ht}\right)^{\beta_1}\right)^{\frac{1}{\beta_2}} + \varepsilon$
(9)	Newnham	$di = d \times \beta_1 \times \left(\frac{\frac{Ht}{Hi}}{Ht - 1.3}\right)^{\beta_2} + \varepsilon$
(10)	Rustagi-Loveless	$di = \beta_1 + \beta_2 \times d \times \left(\frac{\frac{Ht}{Hi}}{Ht - 1.3}\right)^{\beta_3} + \varepsilon$

di = Diameter at different heights above the stem (cm); *d* = Normal diameter (m); *Ht* = Total height (m); *Hi* = Height at different *di* above the stem; β_i = Parameters to be estimated; ε = Error term.

The selection of the model was based on the significance of the parameter (*a*=0.05), the coefficient of determination (R^2, Equation 11), the root mean square error (*RMSE*, Equation 12), the Akaike (*AIC*, Equation 13) and Bayesian (*BIC*, Equation 14) information criterion, and the natural logarithm of the likelihood function ($F(\theta)$, Equation 15) and Bias (Equation 16) (Bronisz & Mehtätalo, 2020; Hernández et al., 2013).

$$R^2 = 1 - \frac{(n-1)\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{(n-p)\sum_{i=1}^{n}(y_i - \acute{y}_i)^2} \qquad (11)$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n-p}} \qquad (12)$$

$$AIC = 2 \times p + n \times nl\left(\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n}\right) \qquad (13)$$

$$BIC = n \times nl\left(\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n}\right) + p \times nl(n) \qquad (14)$$

$$logLik = \prod_{i=1}^{n} f_i \times (y_i|\theta) \qquad (15)$$

$$Bias = \sum_{i=1}^{n}\left(\frac{y_i - \hat{y}_i}{n}\right) \qquad (16)$$

Where:

y_i, $\hat{y}_i$, and $\bar{y}_i$ = Observed, estimated and average values, respectively

n = Total number of data used in model fitting

p = Number of parameters (Table 1)

f_i = Model function

θ = Probability of a parameter vector

nl = Natural logarithm

Once the base expression was selected, it was fitted using the MEM technique with the *nlme* function by maximum likelihood (Pinheiro & Bates, 2000; Pinheiro et al., 2025; R

Core Team, 2024) and the First-Order (FO) expansion method (Fu et al., 2014; Yang & Huang, 2013), additively including the grouping level per individual ($+u_i$) in the value of one or the combination of several of its fixed parameters (β_1) (Zuur et al., 2009). For example, in Clutter's Equation: $di = \beta_1 d^{\beta_2}(Ht - Hi)^{(\beta_3 + u_i)} Ht^{\beta_4} + \varepsilon$ (Equation 17), where the parameter selected for the inclusion of random effects is β_3, the best values of the *AIC*, *BIC* and *logLik* statistics should be included, the *RMSE* and Bias should be the lowest, and all its parameters should be significant (*a*=0.05).

The correction for heteroscedasticity was made by means of the *varExp* expression: $var(\varepsilon_{ij}) = exp^{2 \cdot \delta_1 \cdot v_i}$ (Equation 18), where $var(\varepsilon_{ij})$ is the variance function evaluated in the variance covariate of the predictor residuals (v_i), while δ_1 and δ_2 refer to the coefficients of the variance function, which will be specific for each level (δ_1) (Bronisz & Mehtätalo, 2020; Pinheiro & Bates, 2000). The problems of repeated measures on the same individual were addressed through a self-correlation moving average (ARMA) correlation structure of order (*p*, *q*), where *p* and *q* are the non-negative integers specifying the autoregressive and moving average order of the ARMA structure, respectively, in which case both have the default value of 0 (Pinheiro et al., 2025; Zuur et al., 2009).

Compliance with the regression assumptions was carried out by means of the graphical tests of normality in the frequency of the residuals, the homoscedastic distribution of the residuals and the self-correlation of the errors (Martínez-González et al., 2014). Likewise, the distribution of the random values of the parameters per tree obtained in the MEM adjustment was verified.

The implicit *Vs* equation of all expressions of *di* (Table 1) was determined through the value of its parameters, after which the value of *d* along the stem was integrated into the *Hi* as a solid of revolution; for the *Vt*, the expression 19 was used (Flores et al., 2021; Hernández-Ramos et al., 2018; Pompa et al., 2009; Ramírez-Martínez et al., 2018):

$$Vt = k \int_{H_1}^{H_2} d^2 \, \delta Hi \qquad (19)$$

Where:

Vt = Stem volume

k = Volumetric constant

δ = Differential on the stem height

H_1 and H_2 = Heights that define the integral and that can vary from H_1=0 up to H_2=*Ht*

d = Normal diameter (m)

Estimates of the *Vs* with the implicit models were contrasted through an analysis of the observed values as independent means with a *t*-test at a *p*=0.05 (Infante & Zarate, 2012). The hypotheses were: H_0: there is no difference between the estimates ($\mu_1=\mu_2$), and H_a: the real value of the population mean (*Vs*) is different from the value established by the H_0 ($\mu_1 \neq \mu_2$).

Results

The statistic fit exhibited non-significant parameters (*a*<0.05) in the *Cielito* 1, *Rentería* and Rustagi-Loveless expressions; Amidon's and Newnham's Equations only explained 42.6 and 78.9 % of the sample variability, respectively; while Clutter, *González*, and Forslund had an R^2 value above 0.934. Clutter's expression presented the lowest values in *AIC*, *BIC* and *logLik*, as well as the smallest deviations in *RMSE* and Bias; therefore, it was selected as the base Equation to develop the compatible volume and taper model (Table 2).

Table 2. Parameter values and goodness-of-fit criteria for nonlinear least-squares taper functions.

ID	Parameter	Value	*Se*	*t*-value	*Pr*>|*t*|	Stat.	Value	Stat.	Value
(3)	β_1	2.310	0.21	11.16	<0.001	R^2	0.978	*BIC*	884.1
	β_2	0.978	0.04	27.08	<0.001	*RMSE*	1.401	*logLik*	-428.3
	β_3	1.432	0.03	48.69	<0.001	*AIC*	866.5	*Bias*	3.2E-02
	β_4	-1.670	0.05	-30.87	<0.001				
(4)	β_1	1.205	0.21	5.84	<0.001	R^2	0.973	*BIC*	925.6
	β_2	1.045	0.61	1.70	0.090	*RMSE*	1.539	*logLik*	-451.8
	β_3	0.210	0.43	0.49	0.627	*AIC*	911.6	*Bias*	9.0E-02
(5)	β_1	-1.211	0.24	-4.96	<0.001	R^2	0.951	*BIC*	1 065.4
	β_2	1.204	0.02	68.99	<0.001	*RMSE*	2.067	*logLik*	-524.5
						AIC	1 054.9	*Bias*	9.9E-09
(6)	β_1	-4.640	0.52	-8.96	<0.001	R^2	0.426	*BIC*	1670.6
	β_2	-3.237	0.27	-11.81	<0.001	*RMSE*	7.105	*logLik*	-827.0
						AIC	1 660.1	*Bias*	6.1E-10
(7)	β_1	0.046	0.05	0.93	0.356	R^2	0.973	*BIC*	927.2
	β_2	-0.047	0.21	-0.22	0.823	*RMSE*	1.544	*logLik*	-452.6
	β_3	1.492	0.18	8.46	<0.001	*AIC*	913.2	*Bias*	9.6E-02
(8)	β_1	1.780	0.14	13.15	<0.001	R^2	0.934	*BIC*	1 140.6
	β_2	0.463	0.05	8.76	<0.001	*RMSE*	2.409	*logLik*	-562.1
						AIC	1 130.1	*Bias*	1.1E+00
(9)	β_1	0.799	0.02	51.55	<0.001	R^2	0.789	*BIC*	1 425.6
	β_2	0.401	0.02	23.97	<0.001	*RMSE*	4.309	*logLik*	-704.5
						AIC	1 415.0	*Bias*	-3.9E-01
(10)	β_1	0.277	0.19	1.43	0.153	R^2	0.978	*BIC*	877.3
	β_2	1.002	0.01	88.58	<0.001	*RMSE*	1.395	*logLik*	-427.6
	β_3	1.472	0.04	38.28	<0.001	*AIC*	863.3	*Bias*	-1.8E-10

Se = Standard error; Stat. = Statistic; R^2 = Coefficient of determination; *RMSE* = Root mean square error; *AIC* and *BIC* = Akaike and Bayesian information criterion, respectively; *logLik* = Log likelihood logarithm.

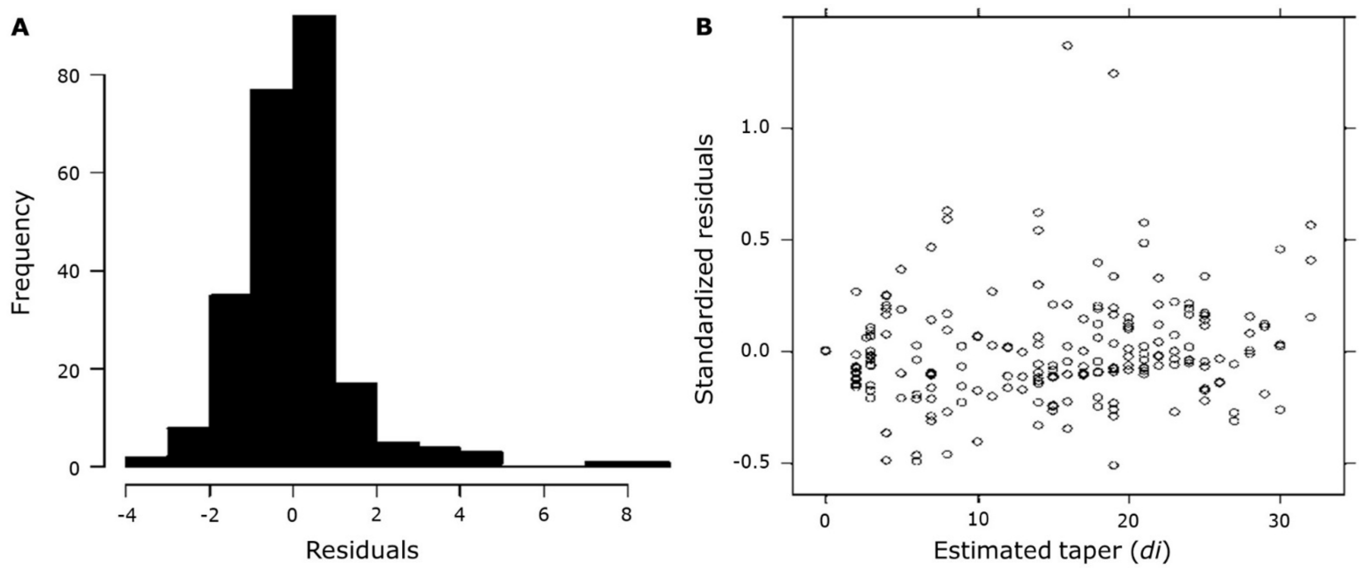
When including the random effect in an additive way per individual, as a grouping level in Clutter's expression, it turned out that when considering the effect in two or more parameters, the values were not significant (a<0.05); whereas, when using the β_1, β_2 and β_4 parameters the values of R^2, *AIC*, *BIC*, *logLik*, *RMSE*, and Bias did not improve the estimates in the *di*; in contrast to when using β_3, which refers to the difference between the *Ht* and *Hi* of each tree. In addition, the MEM adjustment improved the likelihood values by 2.28 % with respect to the NLS fit, while the Bias was reduced by about 56.8 % (Table 3).

Table 3. Parameter values and goodness-of-fit criteria of Clutter's taper model fitted with the mixed-effects modeling technique.

Model	Parameter	Value	*Se*	*t*-value	*Pr*>|*t*|	Stat.	Value	Stat.	Value
(17)	β_1	2.256	0.23	9.99	<0.001	R^2	0.978	*BIC*	873.3
	β_2	0.943	0.04	23.33	<0.001	*RMSE*	1.406	*logLik*	-414.7
	β_3*	1.434	0.03	41.28	<0.001	*AIC*	845.3	Bias	1.38E-02
	β_4	-1.617	0.06	-26.49	<0.001				
	varExp	-0.01	Value of the variance function						
	Phi	0.313	Correlation structure: ARMA(1)						
				SD		Residual			
	Random effects			0.000074		1.63308			

Se = Standard error; R^2 = Coefficient of determination; *RMSE* = Root mean square error; *AIC* and *BIC* = Akaike and Bayesian information criterion, respectively; *logLik* = Log likelihood logarithm; *SD* = Standard deviation; ARMA = Autocorrelation moving average correlation structure; * = Parameter including the random effect; *varExp* = Value of the variance function; *Phi* = Correlation structure: ARMA(1).

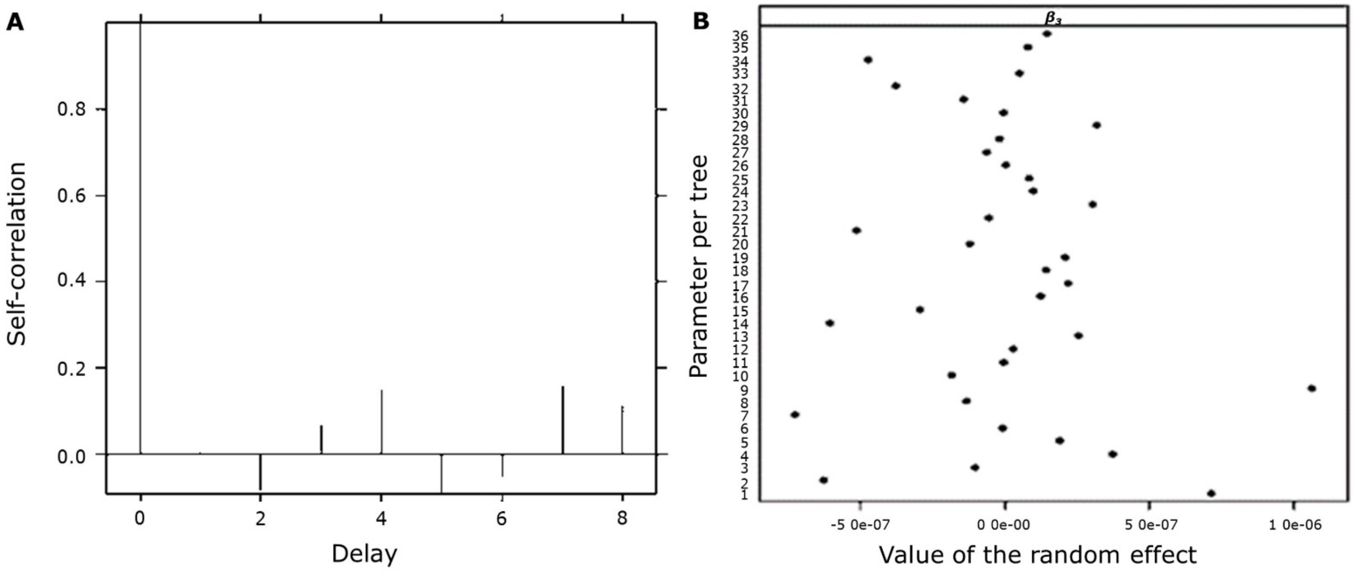
When the regression assumptions were evaluated graphically, a Gaussian bell-shaped frequency distribution of the residuals (Figure 1A), and homoscedastic residuals around zero (Figure 1B) were observed, indicating compliance with normality and lack of heteroscedasticity in the residuals.



A = Normality; B = Homoscedasticity.

Figure 1. Graphical tests of Clutter's taper model fitted using mixed-effects models.

The partial self-correlation values were less than 0.2 (Figure 2A) and indicated that the regression assumption was met in the errors obtained from the fit. In addition, the value of the random effect per individual was plotted within the selected fixed parameter (β_3), which demonstrated the variability of the *di* in the utilized sample (Figure 2B).



A = Autocorrelation of the residuals; B = Distribution of the random values of the parameters tree by tree.

Figure 2. Graphical tests of Clutter's taper model fitted by mixed-effects models.

With the aim of extending the applicability of the above fit, the values of the variance-covariance matrix (vcov) were determined using the MEMs (Table 4), with the idea of a potential calibration with data independent of the sample of the parameter for the difference between *Ht* and *Hi* (β_3), when assessing the variability of *di* in individuals with different characteristics from the information used for the adjustment or outside the geographic area where the FPs of the species under study are developed.

Table 4. Variance-covariance matrix (vcov) of the MEM adjustment of Clutter's taper expression.

vcov	$\boldsymbol{\beta_1}$	$\boldsymbol{\beta_2}$	$\boldsymbol{\beta_3}$	$\boldsymbol{\beta_4}$
β_1	0.05010	-0.00384	0.00034	-0.00463
β_2	-0.00384	0.00161	-0.00001	-0.00132

β_3	0.00034	-0.00001	0.00119	-0.00119
β_4	-0.00463	-0.00132	-0.00119	0.00367

In order to identify the implicit *Vs* models for each *di* expression, mathematical forms were developed to estimate the Schumacher-Hall type *Vs* for Clutter's, *Cielito* 1, and *Rentería* models; *González*, Amidon, and Newnham include in their structure the form of combined variable ($d^2 \times At$), but, overall, the implicit mathematical expression is hardly applicable; the Forslund and Rustagi-Loveless models do not have an explicit mathematical solution, so they are not included in Table 5.

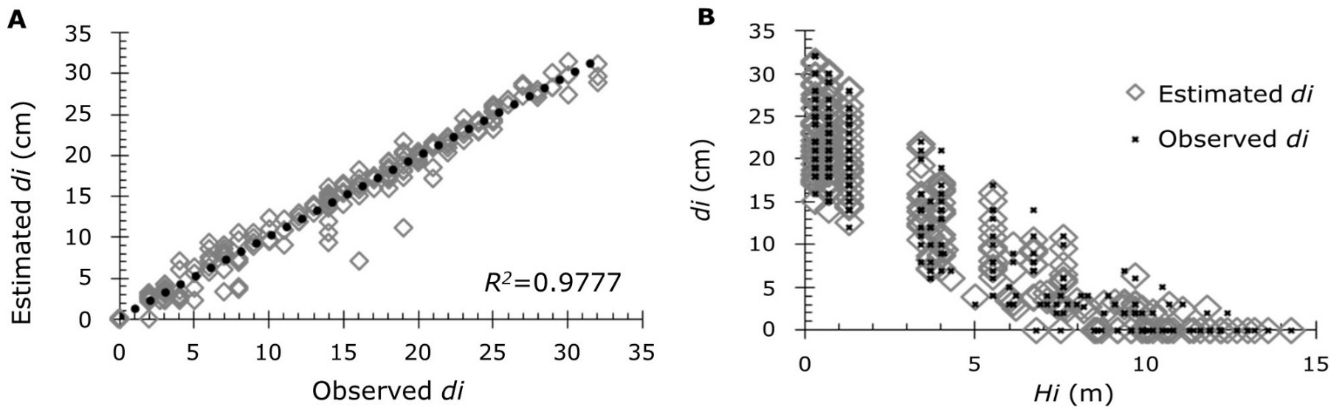
Table 5. Stem volumen models (*Vs*) implicit within the taper functions fitted when considering the normal diameter (*d*, m) and total height (*Ht*, m) as explanatory variables.

ID	Implicit volumen model
(3)	$Vs = 1.3814 \times Ht^{0.5227} \times d^{3.6570}$
(17)	$Vs = 1.3150 \times Ht^{0.6345} \times d^{3.8853}$
(4)	$Vs = 1.0212 \times d^4 \times Ht$
(5)	$Vs = 1.4665(d^2 \times Ht) + 1.4576(d^3 \times Ht) + 0.4829(d^4 \times Ht)$
(6)	$Vs = \frac{1}{(Ht-1.3)^2} \times (d^2 \times Ht(0.7981 \times Ht^4 + 0.1385 \times Ht^3 - 1.2491 \times Ht^2 - 6.3511 \times Ht) + 8.3470 + 1.5383 \times d \times Ht^2 - 1.6183 \times d \times Ht^3 + 5.5047 \times d \times Ht + d^2 \times Ht^2 - 2.6000 \times d^2 \times Ht + 1.6900 \times d^2 - 6.2007 \times d))$
(7)	$Vs = 0.3806 \times d^4 \times Ht$
(9)	$sV = \frac{d^2 \times Ht \left(0.2546 \times d^2 \times Ht^2 \left(\frac{Ht}{Ht-1.3}\right)^{0.9446} + 0.2242 \times d \times Ht^2 \left(\frac{Ht}{Ht-1.3}\right)^{0.9446} + 0.0765 \times Ht^2 - 0.2242 \times d \times Ht \left(\frac{Ht}{Ht-1.3}\right)^{0.9446} \times 1.3 - 0.1530 \times Ht \times 1.3 + 0.1293 \right)}{(Ht-1.3)^2}$

Vs = Stem volumen; *d* = Normal diameter; *Ht* = Total height.

When contrasting the estimates with these implicit models and the observed *Vs*, there were significant differences (*DMS*=0.1313, *F*=84.36; *error*=0.0367, *p*<0.0001). According to Tukey (*a*=0.05), the estimates formed two groups: one in which the implicit models of Clutter, *Rentería*, Amidon and Newnham with respect to the observed data are equal, and the other, with the model of *González*. However, the parameters of the Schumacher-Hall expression (Equation 17) derived from Clutter's *di* function (Equation 3) are compatible and make it possible to arrive at a commercial volume expression (*Vc*), if so desired.

Figure 3 shows a linear tendency between the taper estimates made with Clutter's Equation *versus* the observed values and a description according to the shape of the trees, thereby confirming the explanation of the tendency of the information when describing the profile of the trees established in *Pinus leiophylla* plantations.



A = *Pinus leiophylla* Schiede *ex* Schltdl. & Cham. trees; B = Description of taper versus estimates of *di* at different *Hi* using the Clutter Equation

Figure 3. Comparison of observed diameter (*di*) at different stem sections (*Hi*) versus predicted diameter (taper).

Discussion

Although the presence of non-constant variance in the distribution of residuals (heteroscedasticity) is common when relating biological variables, does not invalidate the estimators obtained by NLS, which continue to be unbiased, they cease to have a minimum variance (Hernández et al., 2013); the MEM approach applied in the present study proved a viable solution for this issue when modeling the *di*. In this sense, Cruz-Cobos et al. (2008) used MEMs to render a linear polynomial model more flexible and increase the predictive capacity of *Cielito* 1 *di* expression for *Pinus cooperi* C. E. Blanco trees in the state of *Durango*, Mexico.

Tamarit et al. (2014) utilized the MEMs and included the random effect within a compatible *di*-*Vc* system, thereby achieving a statistical improvement over NLS, because it allowed control of individual-specific variability; furthermore, the value of the standard error of the model parameters was significantly improved, and the individual estimation Bias was reduced because the MEM technique groups variability specifically by (individual) level, increasing the likelihood in statistic fit of the value of each parameter and reducing the deviations (Pinheiro et al., 2025; Zuur et al., 2009).

The correction of self-correlation errors applied in this study, by including an ARMA-type structure (p=1, q=0), shows satisfactory results, similar to those obtained by Hernández-Ramos et al. (2018) and Flores et al. (2021) when modeling the *di* of clonal trees of the species *Eucalyptus urophylla* S. T. Blake in *Huimanguillo*, state of *Tabasco*, Mexico, and *Pinus pseudostrobus* Lindl. in the *Corona del Rosal ejido*, state of *Nuevo León*, respectively. In both cases, a delay was applied to the residuals (lar1) with a CAR structure (1). The same favorable effect was obtained by Torres et al. (2020) for three provenances of *Pinus caribaea* Morelet var. *hondurensis* (Sénécl.) W. H. Barrett & Golfari and *Pinus elliottii* Engelm. var. *elliottii* Engelm in a FP established in *Las Choapas*, state of *Veracruz*, Mexico, when considering a First-Order autoregressive structure (AR1).

The robustness of the statistic fit cited by Zhang et al. (2021) and demonstrated by Shin et al. (2022) in comparing the classic statistics documented in studies on taper (R^2, *RMSE*, *AIC*, *BIC*, and *logLik*), provides the opportunity to make reliable estimates of taper. In addition, the Equation obtained can be integrated algebraically to obtain its implicit *Vs* or commercial volume (*Vc*) equation describing the distribution of products following the procedure described by McTague and Weiskittel (2021) for mathematical expressions of *di*, such as Clutter's model.

By deriving the compatible trade volume model to fit it, authors like Niño et al. (2018) simultaneously achieve satisfactory results, since they obtain accurate expressions of *Vs* and *Vc*, in addition to modeling the taper. This strategy is also applied by Cruz-Cobos et al. (2008), Hernández-Ramos et al. (2018), and Torres et al. (2020) to model the *di* with different fitting techniques and approaches, thereby generating, in all cases, an expression of *Vs* applicable to the sample of trees analyzed for specific taxa and regions.

The Clutter-type taper model is statistically robust (Shin et al., 2022; Zhang et al., 2021); therefore, it is reliable for describing the forest profile of individuals in FPs of *Pinus leiophylla*. When modeling the *di* using the *Cielito* 2 expression, Ramírez-Martínez et al. (2018) also found the Schumacher-Hall expression to be adequate for predicting the *Vs* in *Pinus ayacahuite* C. Ehrenb. *ex* Schltdl. trees in forests of *Ixtlán de Juárez*, state of *Oaxaca*, Mexico. Furthermore, by using compatible expressions to estimate *di* and *Vs*, estimation biases are operationally reduced, as described by Pompa et al. (2009), with Biging's expression and the implicit expression of the constant morphic coefficient of *Pinus arizonica* Engelm. in Southwestern *Chihuahua* state, Mexico.

The model for the *Vs* of Schumacher-Hall derived from Clutter's taper expression agrees with the expression of Cruz-Cobos et al. (2008) but is derived from the *di* of the *Cielito* 1 model. The results differ from those reported by Torres et al. (2020), who present a complex mathematical form that is not very applicable in practice and which contains in its structure a combined variable expression, similarly to the *Vs* models obtained in this work for the Equations of Amidon (Equation 6) and Newnham (Equation 9) equations (Table 3); also, it contrasts with the model of Pompa et al. (2009) for the *Vs*, which is of the morphic coefficient type and is compatible with Biging's expression.

In general, the fit with the MEM approach presented in this paper overcomes the classical heteroscedasticity and multicollinearity problems in the biological and longitudinal data (Correa & Salazar, 2016; Pinheiro & Bates, 2000; Pinheiro et al., 2025; Zuur et al., 2009). Furthermore, it reduces the deviations in the estimation of the variable of interest with greater certainty through statistical improvement (Cruz-Cobos et al., 2008; Tamarit et al., 2014). However, other strategies and approaches to analysis, such as models based on the biology of the stem shape and growth or quantifying the effect of geographic location of species, have yet to be explored (McTague & Weiskittel, 2021).

The recorded values of the variance-covariance matrix provide the guideline for a further calibration to expand its applicability, as proposed by Çakir and Kahriman (2018) and Saygili and Kahriman (2023), based on the methodology proposed by Şenyurt et al. (2017), Yang et al. (2009), and Zhang et al. (2021), so that it can be used for Clutter's taper model in the β_3 parameter regarding the difference between *Ht* and *Hi*.

Conclusions

The MEM fitting strategy applied to the taper model of *Pinus leiophylla* trees is satisfactory for correcting the problems of heteroscedasticity and error self-correlation. There is also a statistical improvement and a reduction of individual bias in the estimates, in contrast to the Nonlinear least squares adjustment (NLS).

Clutter's taper model and its respective expression of timber volume as per Schumacher and Hall are compatible in their parameters and provide a reliable description of the stem profile of *Pinus leiophylla* trees; therefore, they can be used with certainty in the projection of timber stocks for a differentiated market of the evaluated *P. leiophylla* forest plantations.

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Conflict of interest

The authors declare that they have no conflict of interest.

Contribution by author

H. Jesús Muñoz-Flores: fieldwork, information debugging and analysis, text review and correction; Jonathan Hernández-Ramos: conceptualization of research, information debugging and analysis, text review and correction; Rubén Barrera-Ramírez: fieldwork, information debugging and analysis, manuscript writing; Xavier García-Cuevas: conceptualization of research, information debugging and analysis, text review and correction; Adrián Hernández-Ramos: conceptualization of research, manuscript writing; Martín Gómez-Cárdenas: fieldwork, manuscript writing.

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